MATH 732: CUBIC HYPERSURFACES

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1. Universal Hypersurfaces

These notes are based on [Huy23, §2.1]. Please see the disclaimer section.

A degree d hypersurface $X \subseteq \mathbf{P} = \mathbf{P}^{n+1}$ has equation:

$$\sum a_I x^I = 0$$

where x^I represents a monomial in $x_0, \dots, x_{n+1}$ of degree d. Any non-zero vector (a_I) of coefficients gives rise to such a hypersurface, and two such coefficient vectors (a_I) and (b_I) give the same hypersurface if and only if they are scalar multiples of each other. In other words, we can parametrize all such hypersurface by the projective space:

$$|\mathcal{O}_{\mathbf{P}}(d)| = \mathbf{P}^{N(d,n)} \quad \left(N = N(n,d) = \binom{n+1+d}{d} - 1\right).$$

The *universal hypersurface* $\mathcal{X} \subseteq \mathbf{P}^N \times \mathbf{P}$ is defined by the equation:

$$\mathcal{X} = \left(\sum a_I x^I = 0\right) \subseteq \mathbf{P}^N \times \mathbf{P}.$$

Example 1.1. For example, we see that the universal *degree 2 hypersurface in* $\mathbf{P}^1$ is given by the equation:

$$\left(a_{20}x^2 + a_{11}xy + a_{02}y^2 = 0\right) \subseteq \mathbf{P}^2 \times \mathbf{P}^1$$

where we think of a_{ij} as the coordinates on $\mathbf{P}^2$.

We can think of the equation of the universal hypersurface *intrinsically*. If $\mathbf{P} = \mathbf{P}(V)$, then $\mathrm{H}^0(\mathbf{P}, \mathcal{O}(1)) = V$ (from the perspective that $\mathbf{P}$ parametrizes quotients of V). Then $\mathbf{P}^N = \mathbf{P}(\mathrm{Sym}^d(V^\vee))$. And thus:

$$\mathrm{H}^0(\mathbf{P}^N \times \mathbf{P}, \mathcal{O}(1,d)) \simeq \mathrm{Sym}^d(V^\vee) \otimes \mathrm{Sym}^d(V)$$

has a canonical section.

The universal hypersurface is smooth! If we project $\mathcal{X} \to \mathbf{P}$, this is actually a projective bundle. Which projective bundle? The fibers over a point $p \in \mathbf{P}$ are precisely the degree d hypersurfaces through that point. So these hypersurfaces correspond to the projective space $\mathbf{P}(\mathrm{H}^0(\mathbf{P}, \mathfrak{m}_p(d)))$. We can globalize this perspective. The kernel of the evaluation map:

$$0 \to \mathcal{K}_d \to H^0(\mathbf{P}, \mathcal{O}(d)) \otimes_k \mathcal{O}_{\mathbf{P}} \xrightarrow{ev} \mathcal{O}(d) \to 0$$

has the correct kernel at each point, so we have $\mathbf{P}(K_d) = \mathcal{X}$.

There are lots of isomorphic hypersurfaces. The easiest way (and almost the only way) to produce isomorphic hypersurfaces is to look at the image of a hypersurface under an element $A \in \mathrm{PGL}(n+2) = \mathrm{Aut}(\mathbf{P})$. Without defining terms at the moment, the *moduli space* of degree d hypersurfaces is the (GIT) quotient:

$$M(d,n) := |\mathcal{O}_{\mathbf{P}}(d)|_{sm} // \mathrm{PGL}(n+2).$$

Morally the moduli space parametrizes degree d hypersurfaces up to isomorphism. The dimension is computable:

$$\dim M(d,n) = \binom{n+1+d}{d} - (n+2)^2.$$

For cubics of small dimension this gives:

n	0	1	2	3	4	5
$\dim M(3,n)$	0	1	4	10	20	35

Mostly we care about the Zariski open set of smooth cubics:

$$U(d,n) = \mathbf{P}^{N(n,d)} \setminus D(d,n) \subseteq \mathbf{P}^{N(n,d)}.$$

The singular cubics are parametrized by the *discriminant locus.*

Theorem 1.2. *The discriminant locus $D(d,n) \subseteq \mathbf{P}^{N(n,d)}$ is an irreducible divisor of degree $(d-1)^{n+1} \cdot (n+2)$. For cubics, this gives:*

$$\deg D(3,n) = 2^{n+1}(n+2).$$

Example 1.3. Before working out the general case, let's figure it out for the universal quadratic equation. In this case, it is nice to think of the universal quadratic form as being associated to a bilinear form with associated matrix:

$$A = \begin{bmatrix} a_{20} & \frac{1}{2}a_{11} \\ \frac{1}{2}a_{11} & a_{02} \end{bmatrix}.$$

A pair of points in $\mathbf{P}^2$ is singular $\iff$ it's a double point

$$\iff a_{20}x^2 + a_{11}xy + a_{02}y^2 = (\sqrt{a_{20}}x + \sqrt{a_{02}}y)^2 \iff \det(A) = 0.$$

Now $\det(A)$ has degree 2, which equals $(2-1)^1 \cdot (0+2)$.

Proof. We give a proof when the characteristic does not divide d. By the Jacobian criterion, a hypersurface $X_{(a_I)}$ is singular if and only if the intersection:

$$X_{(a_I)} \cap \left(\cap_{i=0}^{n+1} V_i\right) = X_{\text{sing}} \neq \varnothing$$

where V_i is defined by the equation:

$$V_i = \left(\partial_{x_i} F = \sum a_I \frac{\partial x^I}{\partial x_i} = 0\right) \subseteq \mathbf{P}.$$

Alternatively, we can think of these as hypersurfaces:

$$\mathcal{V}_i \subseteq \mathbf{P}^{N(n,d)} \times \mathbf{P}$$

of degree $(1, d-1)$. From this, the singularities of the fibers of the universal hypersurface are given by:

$$\mathcal{X} \cap \left(\cap_{i=0}^{n+1} \mathcal{V}_i\right) = \mathcal{X}_{\text{sing}} \subseteq \mathcal{X} \subseteq \mathbf{P}^{N(n,d)} \times \mathbf{P}.$$

So we want to compute $\mathcal{X}_{\text{sing}}$ and its image $D(d,n) \subseteq \mathbf{P}^{N(n,d)}$.

First, we note that $\cap \mathcal{V}_i \subseteq \mathcal{X}$ and so $\mathcal{X}_{\text{sing}} = \cap \mathcal{V}_i$. This is a consequence of the *Euler equation.* If F is the equation defining the universal hypersurface:

$$dF = \sum (\partial_{x_i} F) x_i. \qquad \text{(Euler equation)}$$

So if all $\partial_{x_i} F = 0$ (and the characteristic does not divide d), then $F = 0$.

Second, the singular locus $\mathcal{X}_{\text{sing}} \subseteq \mathbf{P}^{N(n,d)} \times \mathbf{P}$ is actually smooth (and is a projective bundle). Again, we look at the hypersurfaces through a point $p \in \mathbf{P}$, but only consider the ones singular at that point. This corresponds to looking at the kernel of the map of sheaves:

$$\mathrm{H}^0(\mathbf{P}, \mathcal{O}(d)) \otimes_k \mathcal{O}_{\mathbf{P}} \to \mathcal{O}_{\mathbf{P}}(d-1)^{\oplus n+2} \quad (F \mapsto \oplus \partial_{x_i} F).$$

This map is surjective, so the kernel is a vector bundle, whose projectivization gives $\mathcal{X}_{\text{sing}}$. The dimension of $\mathcal{X}_{\text{sing}}$ is thus $N(d,n) - 1$, and we have it is smooth and irreducible. It follows that $D(d,n)$ is irreducible of dimension *at most* $N(d,n) - 1$.

By the projection formula, for a map of projective varieties $f\colon Y \to Z$ and a Cartier divisor D on Z we have:

$$\deg(f^*(D)^{\dim(Y)}) = \deg(\deg(Y \to f(Y)) f(Y) \cdot D^{\dim(Y)}).$$

(So it can only be non-zero if the map is generically finite onto its image).

We apply this to the map $\mathcal{X}_{\text{sing}} \to \mathbf{P}^{N(n,d)}$. Let h_1 represent the pull back of the hyperplane class on $\mathbf{P}^{N(n,d)}$ (respectively h_2 the pull back of the hyperplane class from $\mathbf{P}$). This corresponds to the line bundle $\mathcal{O}(1,0)$ on $\mathbf{P}^{N(n,d)} \times \mathbf{P}$. We want to compute

$$\deg(h_1^{N(n,d)-1} \cdot \mathcal{X}_{\text{sing}}) = \deg(h_1^{N(n,d)-1} \cdot (\cap \mathcal{V}_i))$$

The divisors $\mathcal{V}_i$ are all of type $(1, d-1)$. So we want to compute

$$\begin{aligned} h_1^{N(n,d)-1} \cdot (h_1 + (d-1)h_2)^{(n+2)} &= h_1^{N(n,d)-1} \cdot \sum \binom{n+2}{i} (d-1)^i h_1^{(n+2-i)} h_2^i \\ &= \binom{n+2}{n+1} (d-1)^{(n+1)} h_1^{N(n,d)} h_2^{(n+1)} \\ &= (n+2) \cdot (d-1)^{(n+1)}. \end{aligned}$$

So this shows that the map $\mathcal{X}_{\mathrm{sing}} \to D(d,n)$ is *finite* (which proves it is a divisor). To complete the proof we need to show the degree of the map is one. At least if the map is separable (let's assume characteristic 0), this amounts to checking that the *generic singular hypersurface* has only one singularity.

If the generic singular hypersurface has multiple singular points, then for a generic point $p \in \mathbf{P}$, the generic hypersurface that is singular at p will be singular at another point in $\mathbf{P}$. However, by Bertini's theorem (with base points), a generic hypersurface that is singular at p is smooth everywhere else. □

Remark 1.4. A general, singular hypersurface has exactly one singularity which is an ordinary double point. As a consequence, a *generic pencil* of degree d hypersurfaces consists of only smooth hypersurfaces and singular hypersurfaces with at most 1 ordinary double point. Such a pencil is called a *Lefschetz pencil*.

Exercise 1. *Filling in some things from class:*

(1) *Prove the map of sheaves*

$$\mathrm{H}^0(\mathbf{P}, \mathcal{O}(d)) \otimes_k \mathcal{O}_{\mathbf{P}} \to \mathcal{O}_{\mathbf{P}}(d-1)^{\oplus n+2} \quad (F \mapsto \oplus \partial_{x_i} F)$$

is surjective.

(2) *Prove that for any $p \in \mathbf{P}$, the generic hypersurface that is singular at p has an ordinary double point at p (i.e. the tangent cone at p is a non-singular quadratic form).*

Exercise 2. *As we mentioned in our example, the discriminant locus $D(2,n)$ of the universal quadratic form corresponds to those whose associated bilinear form is singular (i.e. has nullity ≥ 1).*

(1) *Reprove the theorem on the degree of the discriminant for these forms.*
(2) *Prove that the singular locus of $D(2,n)$ corresponds to the set of bilinear forms whose matrix has nullity ≥ 2.*
(3) *(Optional) Can you figure out what happens in characteristic 2?*

References

[Huy23] Daniel Huybrechts. *The geometry of cubic hypersurfaces*, volume 206 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2023.